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<u>Spontaneous stochasticity of velocity in turbulence models</u>

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Spontaneous stochasticity of velocity in turbulence models

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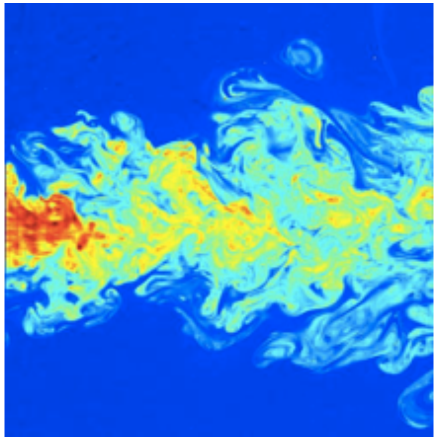


Model

3D Navier-Stokes turbulence problem for **large Reynolds numbers**

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla P + \nu \Delta \mathbf{u}, \quad \nabla \cdot \mathbf{u} = 0$$

Gledzer shell model of turbulence (GOY/Sabra for imaginary speeds)

$$\frac{du_n}{dt} = \left(\frac{1}{2} k_{n-1} u_{n-1} u_{n-2} + \frac{1}{2} k_n u_{n+1} u_{n-1} - k_{n+1} u_{n+2} u_{n+1} \right) - \nu k_n^2 u_n$$

Shells n = 1,2,... describe speed fluctuations u_n at wavenumbers $k_n = 2^n$

Boundary conditions ("forcing"): $u_0 = const, \quad u_{-1} = const$

Energy: $E = \sum u_n^2$ Enstrophy: $\Omega = \sum k_n^2 u_n^2$ Helicity: $H = \sum (-1)^n k_n u_n^2$

Energy flux: $\Pi_n = k_n u_{n-1} u_n u_{n+1} + 2 k_{n+1} u_n u_{n+1} u_{n+2}$

Energy balance: $\frac{dE}{dt} = \Pi_0 - 2\nu\Omega$

Stationary solutions

Model symmetries

$$u_n \mapsto 2u_{n+1}, \quad \nu \mapsto 2^2\nu;$$

$$u_n \mapsto cu_n, \quad \nu \mapsto c\nu, \quad t \mapsto t/c;$$

$$u_n \mapsto \sigma_n u_n, \quad \sigma_n = \pm 1, \quad \sigma_n\sigma_{n+1}\sigma_{n+2} = 1,$$

$$t \mapsto t + t_0.$$

Stationary solutions of inviscid model (Biferale et al. 1995)

$$u_{n+2}u_{n+1} = \frac{1}{4}u_{n+1}u_{n-1} + \frac{1}{8}u_{n-1}u_{n-2}, \quad n = 1, 2, \ldots.$$

$$c_{n+2} = \frac{1}{2} + \frac{1}{2c_{n+1}}, \quad c_n = \frac{2u_n}{u_{n-3}},$$

Fixed-point attractor: $c_n = 1$ as $n \to \infty$.

Period-3 asymptotic form of stationary solutions: $u_{3n+i} \to a_i k_{3n+i}^{-1/3}, \quad i = 1, 2, 3$

(Kolmogorov scaling law / shock wave)

Viscous stationary solutions

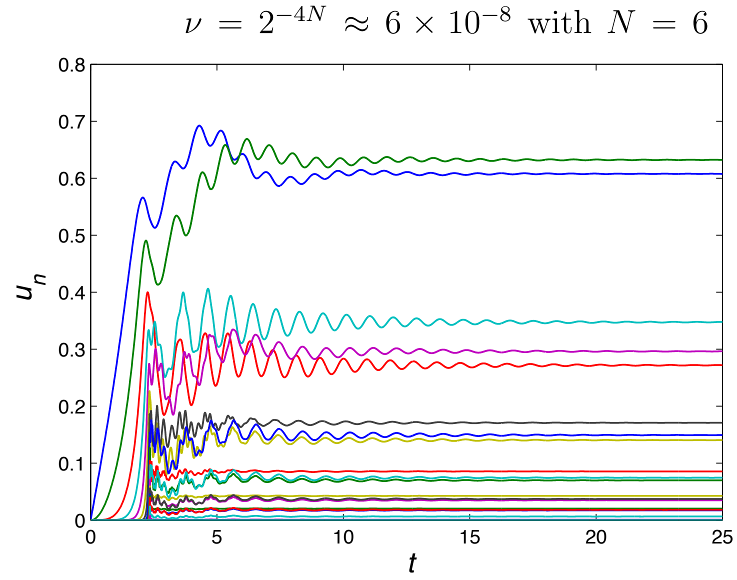
$$u_{n+2}u_{n+1} = \frac{1}{4}u_{n+1}u_{n-1} + \frac{1}{8}u_{n-1}u_{n-2} - \frac{1}{2}\nu k_n u_n$$

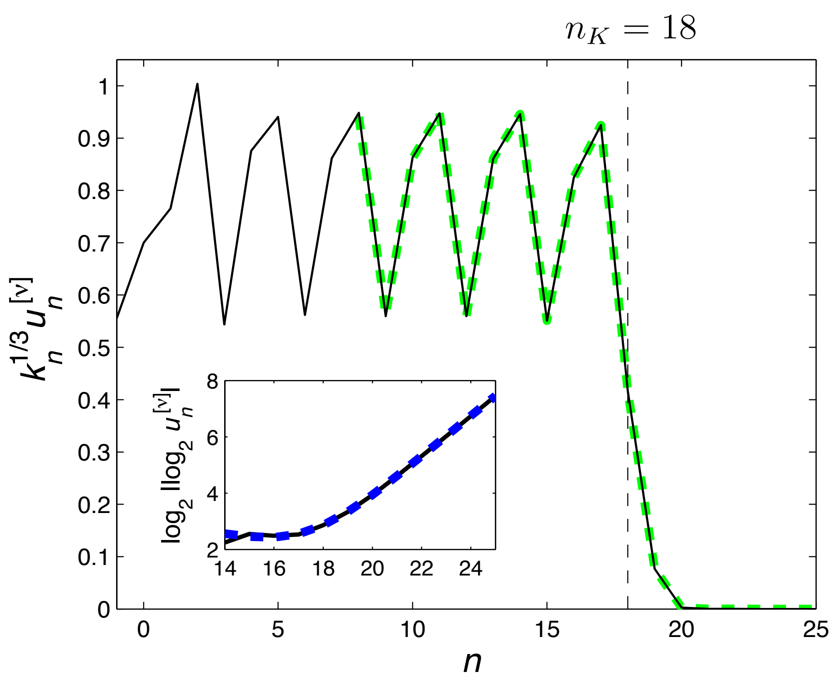
Kolmogorov wavenumber: $k_K \sim \nu^{-3/4}$

Kolmogorov shell number: $n_K = \log_2 k_K = -\frac{3}{4}\log_2 \nu$

Inviscid dynamics: $n \ll n_K$

Viscous range: $n \gtrsim n_K$





Viscous range asymptotic

Dominant terms: $u_{n-1}u_{n-2} \approx 4\nu k_n u_n.$

$$\log_2|u_n| \approx \log_2|u_{n-1}| + \log_2|u_{n-2}| - 2 - n - \log_2 \nu$$

Solution: $\log_2|u_n| \approx b\sigma^n + \tilde{b}\tilde{\sigma}^n + 5 + n + \log_2 \nu$

$$\sigma = (1+\sqrt{5})/2,\ \tilde{\sigma} = (1-\sqrt{5})/2$$

Asymptotic: $|u_n| \approx 32\nu k_n 2^{b\sigma^n}, \quad n \gg n_K$

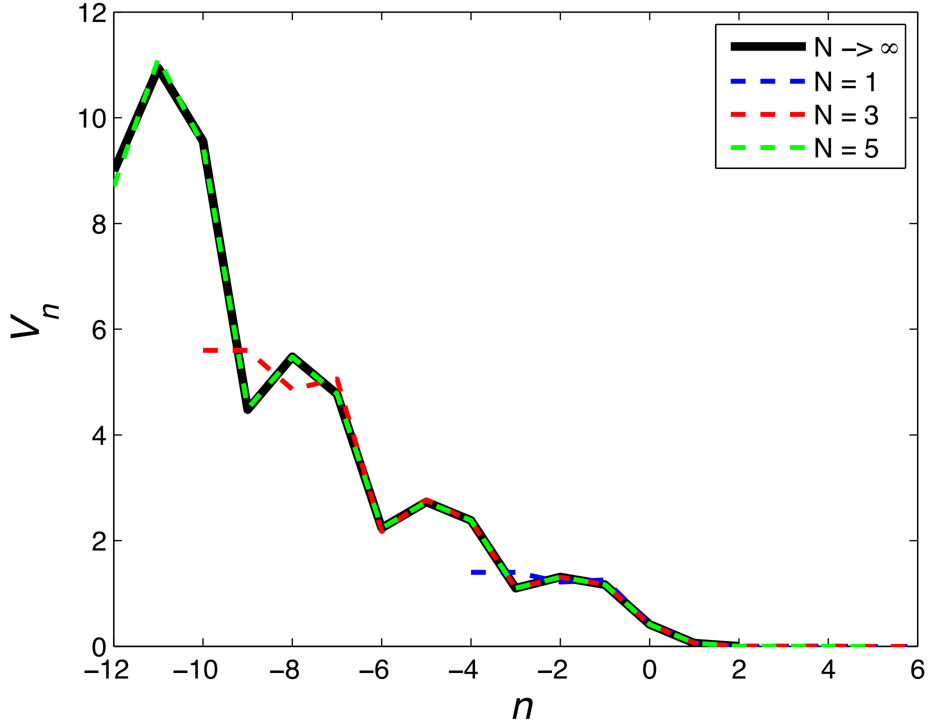
Inviscid limit for viscous range

Model symmetry: $u_n \mapsto 2^{1/3} u_{n+1}, \quad \nu \mapsto 2^{4/3}\nu, \quad t \mapsto 2^{2/3} t$

Stationary solution symmetry (integer N): $u_n \mapsto 2^N u_{n+3N}, \quad \nu \mapsto 2^{4N}\nu. \quad (n_K \mapsto n_K)$

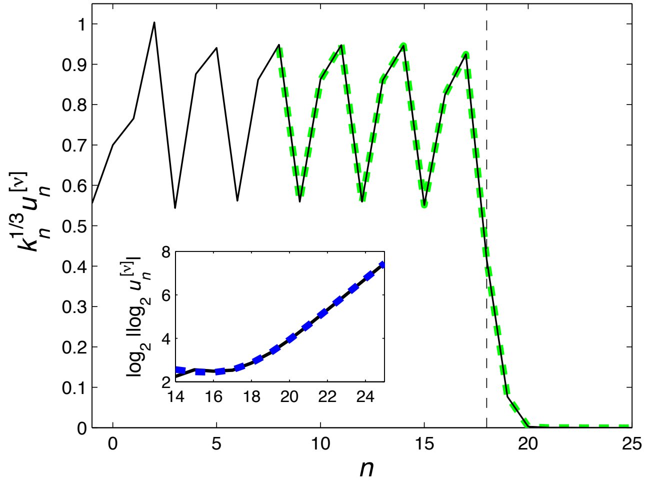
Limit of vanishing viscosity (for fixed boundary conditions and any fixed parameter χ):

$$V_n(\chi) = \lim_{N\to\infty} 2^N u_{n+3N}^{[\nu_N]}, \quad \nu_N = 2^{-4(\chi+N)}, \quad n, N \in \mathbb{Z}$$



Infrared asymptotic of the limiting stationary solution V_n

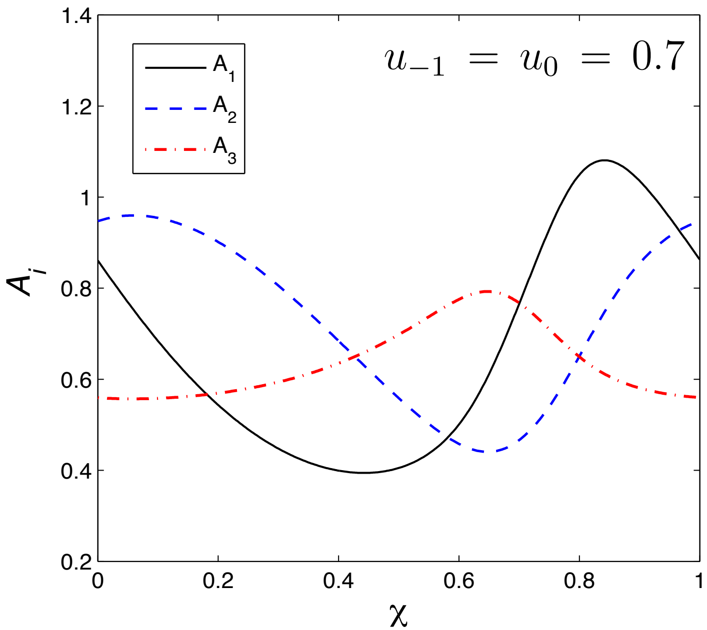
$$V_{3n+i}(\chi) \to A_i(\chi) k_{3n+i}^{-1/3} \quad \text{as} \quad n \to -\infty, \quad i = 1, 2, 3.$$



$$A_i(\chi) = \lim_{n\to-\infty} \lim_{N\to\infty} k_m^{1/3} u_m^{[\nu_N]},$$

$$m = 3(n+N) + i, \quad \nu_N = 2^{-4(\chi+N)}, \quad i = 1, 2, 3.$$

Periodicity: $A_i(\chi + k) = A_i(\chi), \quad i = 1, 2, 3, \quad k \in \mathbb{Z},$



Period-3 solution:

$$U_n(\chi) = \lim_{N\to\infty} u_n^{[\nu_N]}, \quad \nu_N = 2^{-4(\chi+N)},$$

$$U_{3n+i}(\chi) \to A_i(\chi) k_{3n+i}^{-1/3}, \quad i = 1, 2, 3, \quad n \to \infty$$

Universal form of the period-3 solution

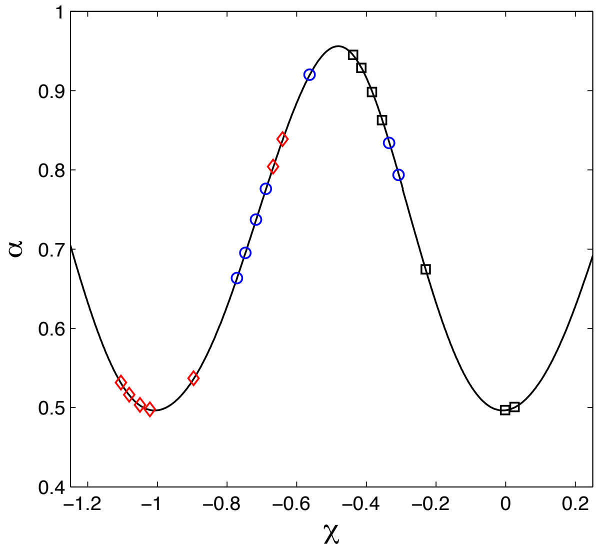
Energy flux (dissipation rate): $D(\chi) = \lim_{n\to\infty} \Pi_n = 3A_1(\chi)A_2(\chi)A_3(\chi)$

Model symmetry: $u_n \mapsto cu_n, \quad \nu \mapsto c\nu, \quad t \mapsto t/c$

Scaling of period-3 solution and energy dissipation rate: $A_i \mapsto cA_i, \quad D \mapsto c^3 D$

$$c = 2^{4(\chi-\tilde{\chi})} \quad \longrightarrow \quad A_i = D^{1/3}\alpha_i\left(\chi + \frac{\log_2 D}{12}\right)$$

Numerical simulations:



$$\alpha_i(\chi) = \alpha\left(\chi - \frac{i}{3}\right), \quad i = 1, 2, 3$$

(complies with the model symmetry)

Universal asymptotic:

$$U_{3n+i} \to a_i k_{3n+i}^{-1/3},$$

$$a_i(\chi, D) = \sigma_i D^{1/3}\alpha\left(\chi - \frac{i}{3} + \frac{\log_2 D}{12}\right), \quad i = 1, 2, 3, \quad n \to \infty$$

Non-stationary solutions

Inviscid limit: definitions

Inviscid limit: $\nu_N = 2^{-4(\chi+N)} \to 0$

Relaxation time for period-3 solution (scaling symmetry): $t_{rel} \propto 2^{-2N} \to 0$

Instantaneous relaxation in inviscid limit!

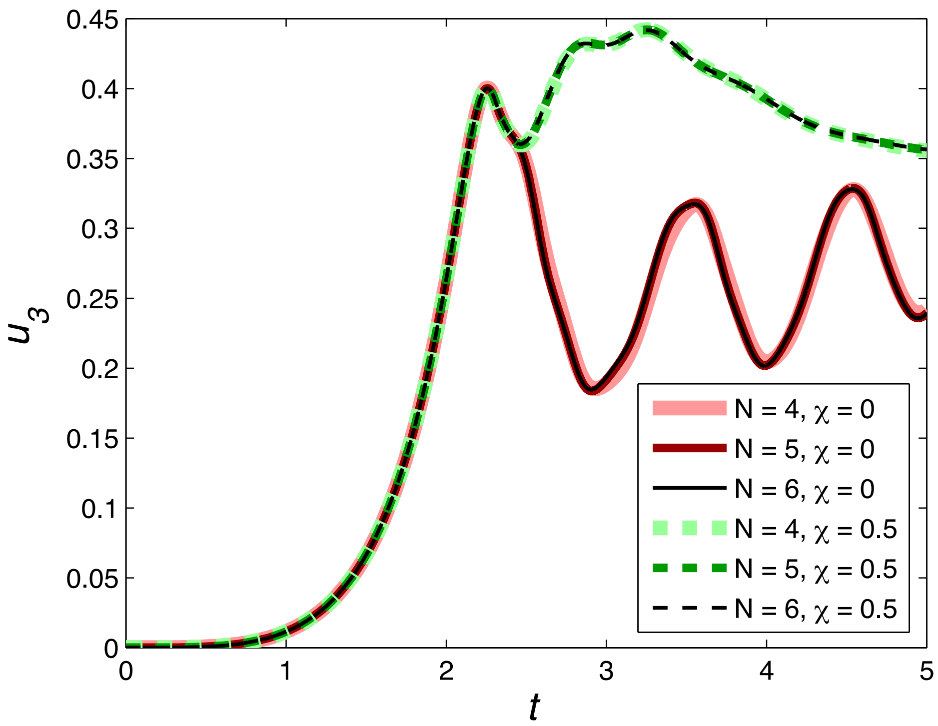
Inviscid limit for **time-dependent** solutions:

$$u_n(t,\chi) = \lim_{N\to\infty} u_n^{[\nu_N]}(t),$$

Ultraviolet limit:

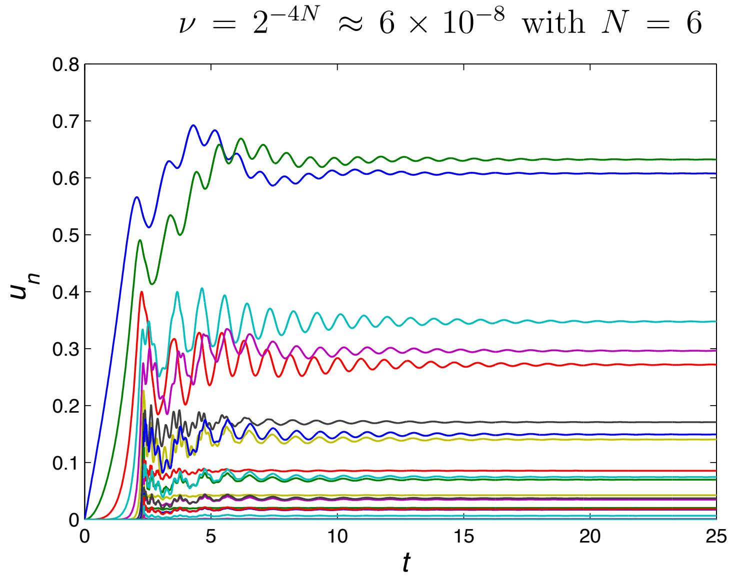
$$u_{3n+i}(t,\chi) \to a_i(\chi, D(t))k_{3n+i}^{-1/3},$$

$$i = 1,2,3, \quad n \to \infty$$

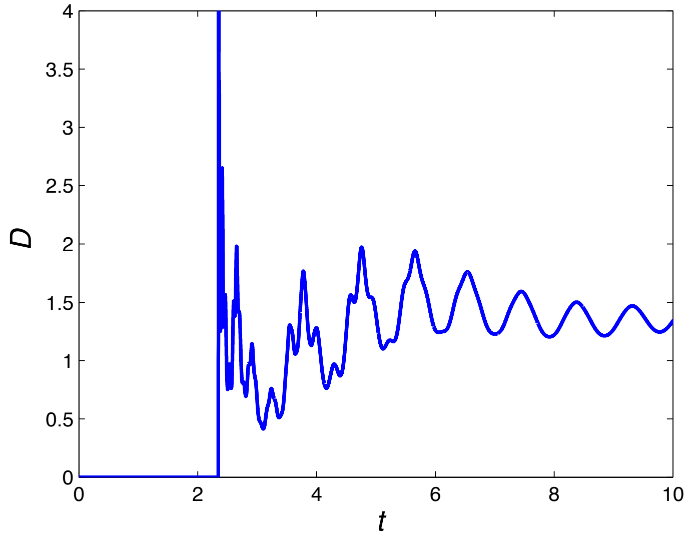


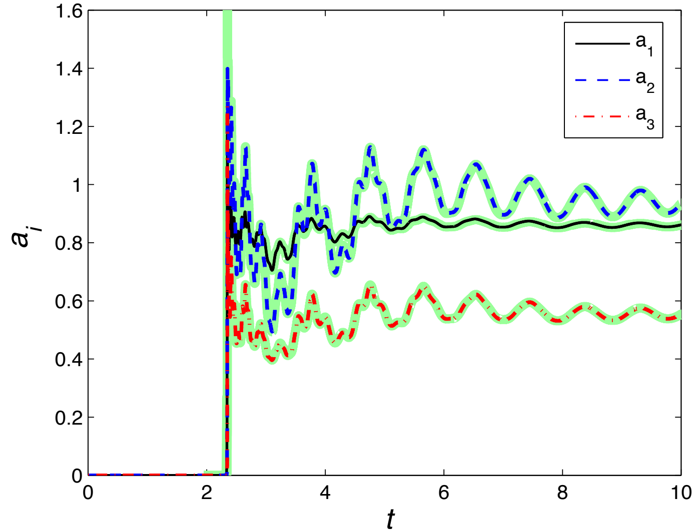
Numerical simulations

Simulation for zero initial and constant boundary conditions:



Ultraviolet asymptotic:

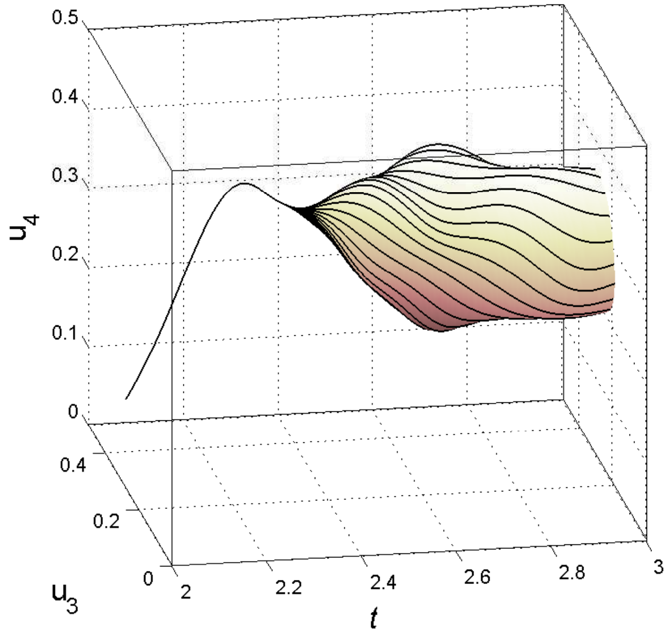




Stochastic definition of limiting solution

$$U_n(t) = \lim_{\mu \to +0} u_n^{[\nu]}(t), \quad \nu = \mu 2^{-4X}$$

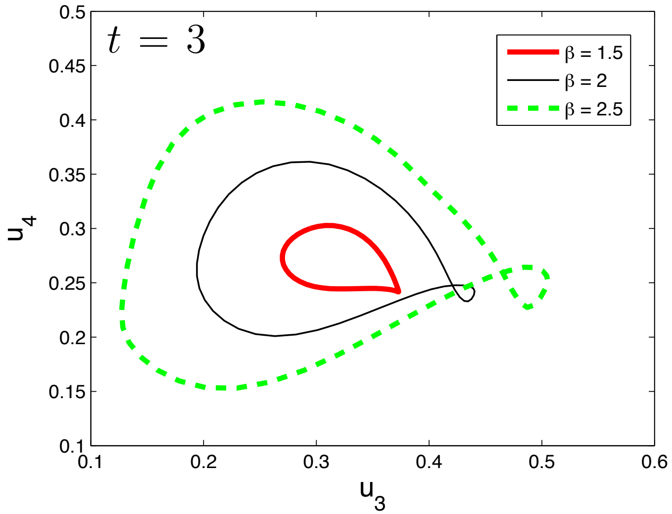
with a random variable X, e.g., uniformly distributes in the interval $[0, 1]$.



Spontaneous stochasticity

Dependence on viscosity **model**:

$$-\nu k_n^{\beta} u_n$$

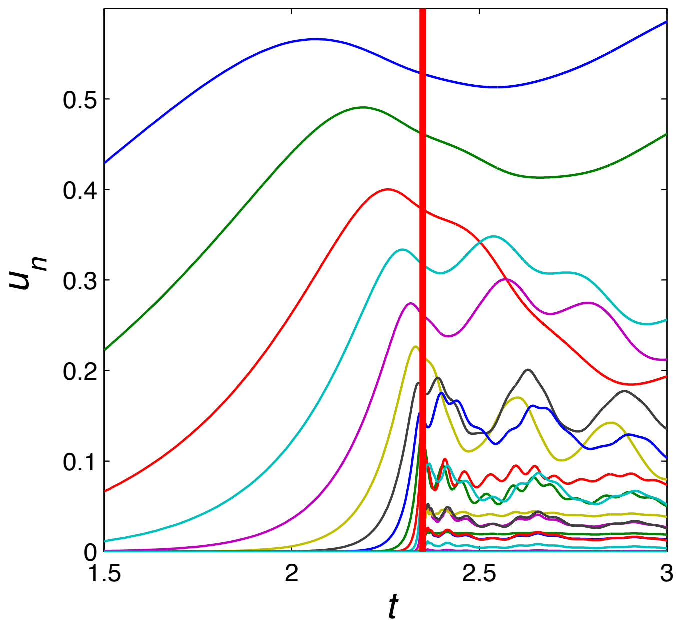


Onset of spontaneous stochasticity: before and after blowup

Left asymptotic of blowup

Universal self-similar asymptotic (Dombre&Gilson, 1998)

$$u_n(t) \to \sigma_n c k_n^{-y} U(c\xi) \quad \text{with} \quad \xi = k_n^{1-y}(t - t_b) \le 0$$



Time scaling:

$$t = t_b + \xi k_n^{y-1} \to t_b^-, \quad n \to \infty$$

Scaling at blowup time:

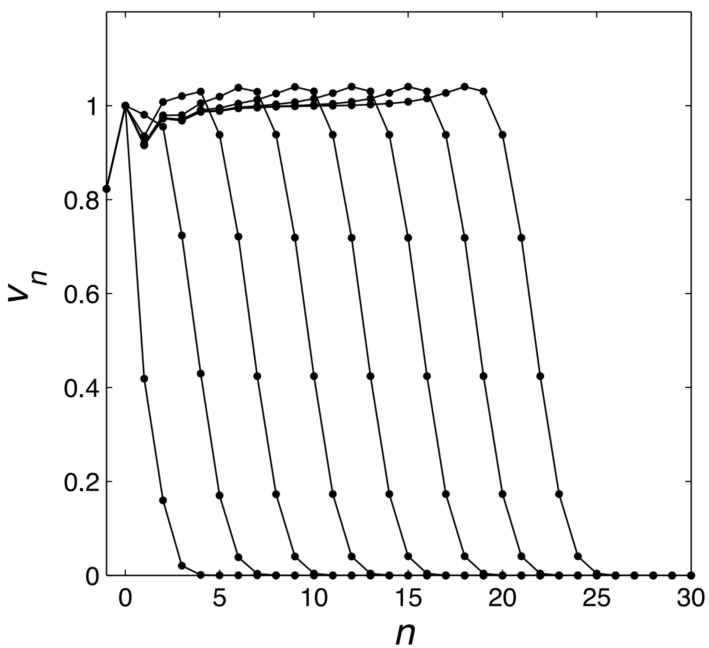
$$u_n(t_b) \to \sigma_n c k_n^{-y} \quad \text{as} \quad n \to \infty$$

Renormalization and traveling wave representation:

$$u_n = \sigma_n c k_n^{-y} v_n, \quad t = t_b - 2^{-\tau}/c$$

$$v_n(\tau) \to V\left(n - \frac{\tau}{\tau_0}\right), \qquad \tau_0 = 1 - y \approx 0.719.$$

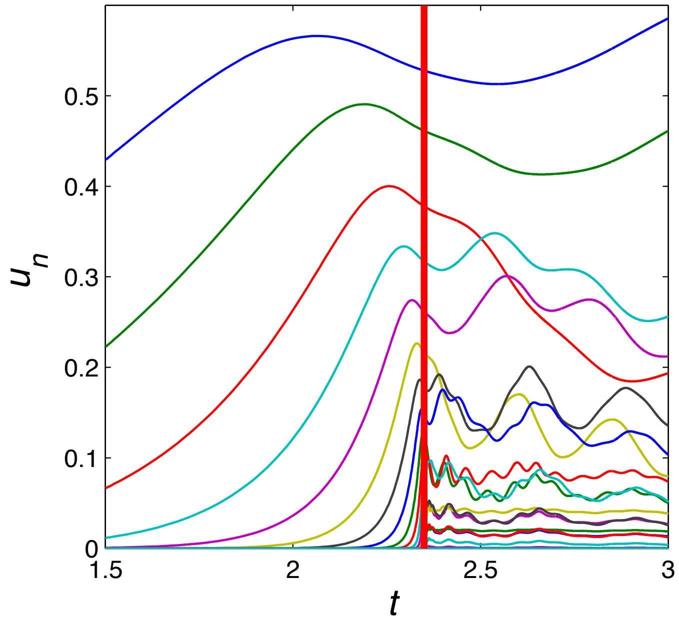
$$\tau = \tau_0 n + const \to \infty, \quad n \to \infty$$



Right asymptotic of blowup: renormalization

Symmetry:

$$u_n \mapsto \sigma_{n-1}\sigma_n 2^{-y} u_{n-1}, \quad t - t_b \mapsto 2^{y-1}(t - t_b), \quad \nu \mapsto 2^{-(1+y)}\nu$$



Renormalization:

$$u_n = \sigma_n c k_n^{-y} w_n, \quad t = t_b + 2^{-\tilde{\tau}}/c,$$

Solution representation:

$$w_n(\tilde{\tau}, \chi) = W_n\left(n - \frac{\tilde{\tau}}{\tau_0}, \chi - \chi_c - \frac{\tilde{\tau}}{\tau_1}\right),$$

$$\chi_c = -\frac{1}{4}\log_2 c, \quad \tau_1 = \frac{4-4y}{1+y} \approx 2.245,$$

Symmetry for renormalized variables:

$$w_n \mapsto w_{n-1}, \quad \tilde{\tau} \mapsto \tilde{\tau} + \tau_0 \qquad \chi \mapsto \chi + \chi_0, \quad \chi_0 = \frac{1+y}{4} = \frac{\tau_0}{\tau_1}$$

$$W_n(\eta_1, \eta_1) \mapsto W_{n-1}(\eta_1, \eta_2)$$

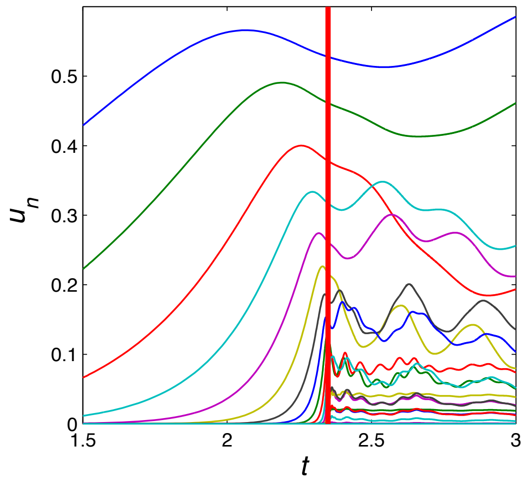
Right asymptotic: universal quasi-periodic solution

Quasi-periodic asymptotic expression independent of *n*:

$$w_n(\tilde{\tau}, \chi) \to W\left(n - \frac{\tilde{\tau}}{\tau_0}, \chi - \chi_c - \frac{\tilde{\tau}}{\tau_1}\right)$$

$$\tau_0 = 1 - y \approx 0.719$$

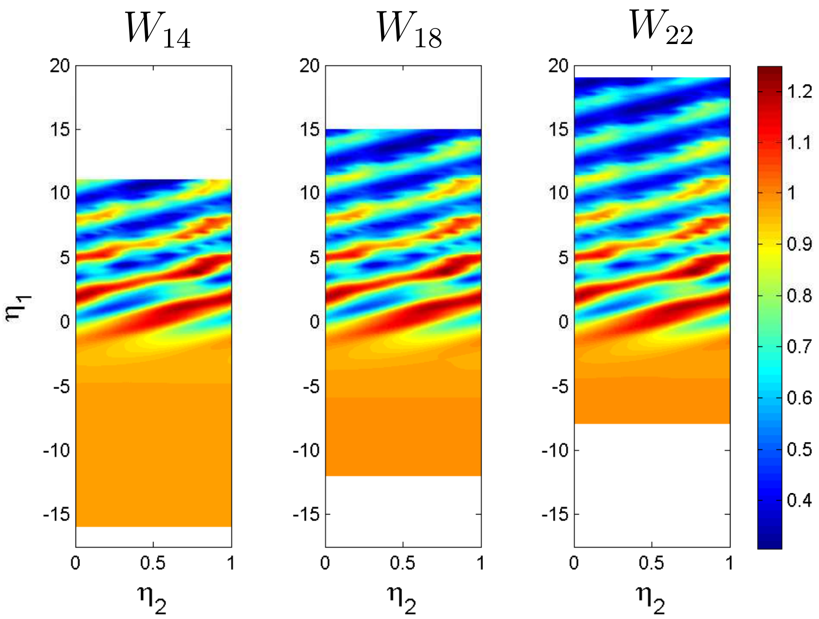
$$\tau_1 = \frac{4 - 4y}{1 + y} \approx 2.245$$

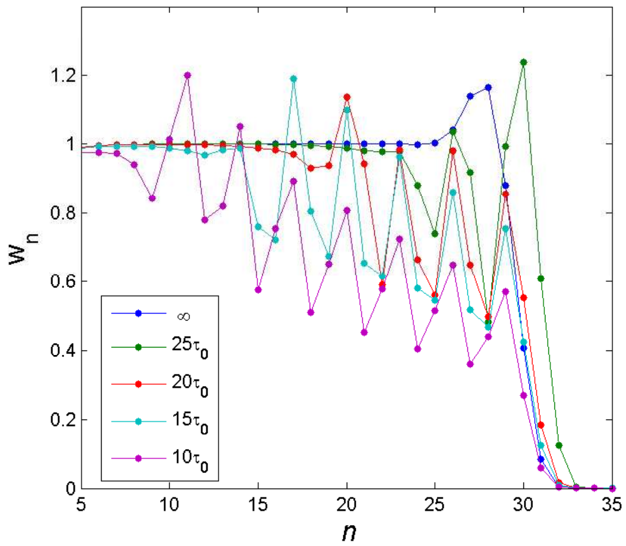


Properties:

$$W(\eta_1, \eta_2) = W(\eta_1, \eta_2 + 1)$$

$$\lim_{\eta_1 \to -\infty} W(\eta_1, \eta_2) = 1 \qquad \lim_{\eta_1 \to \infty} W(\eta_1, \eta_2) = 0$$





Conclusion

- Inviscid limit in the Gledzer shell model is not unique.
- Infinite number of limiting solutions are obtained for vanishing viscosities considered as geometric sequences: powers of 1/16.
- Characterization of limiting solutions is carried out by renormalization of the viscous range. This leads to universal period-3 ultraviolet condition at every time, which depends on the energy dissipation rate.
- Non-uniqueness = spontaneous stochasticity starts at blowup time. Its initial stage has the universal quasi-periodic form.